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# (2, 0) Heterotic Superstring with Wess–Zumino Coupling

**Bang-qing Xu, Peng Xu,[1] Dianyan Xu,[2] and Chang-gui Shao[3]**

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By means of the identity operator and vertex operator technique, reparametrization invariance and BRST symmetry are proven for the heterotic string with Wess–Zumino term coupling the fiber bundle. The motion space of the string is assumed to be a direct product $M_d \otimes G$ of a Minkowski space $M_d$ of dimension $d$ with an intrinsic group manifold $G$ of dimension $d_G$, and turns out to give the critical dimension.

## 1. INTRODUCTION

By means of the conformal field theory, Witten (1986*a,b*) developed a new mathematical framework in noncommutative differential geometry, associated with the derivative given by the BRST charge $Q$. Then, using the method of the interacting vertex of midpoint type, he constructed a superstring. An integration of the string function and an analog of the wedge product $\star$ have been used to form an interaction of Chern–Simons type $\oint \Phi \star \Phi \star \Phi$. The axioms obeyed by the system of $\int$, $\star$, and $Q$ are as follows:

$$\int QW = 0, \qquad \int \Psi = \langle I|\Psi\rangle \tag{1.1a}$$

$$\int \Psi_1 \star \Psi_2 \star \psi_3 = \langle V_3|\Psi_1|\Psi_2|\Psi_3\rangle \tag{1.1b}$$

$$Q(A \star B) = (QA) \star B + (-)^A A \star (QB) \tag{1.1c}$$

$$(A \star B) \star C = A \star (B \star C) \tag{1.1d}$$

[1]China Institute of Atomic Energy, P.O. Box 275, Beijing, China.
[2]Center of Theoretical Physics, Chinese Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing, China, and China Department of Computer Science and Technology, Peking University, Beijing, 100871, China (address for correspondence).
[3]Department of Physics, Hubei University, 430062, Wuhang, China.

The action

$$S = \int \left( \Psi \star Q\Psi + \frac{2}{3} \Psi \star \Psi \star \Psi \right) \tag{1.2}$$

is then invariant under the gauge transformation

$$\delta\Psi = Q\varepsilon - \varepsilon \star \Psi + \Psi \star \varepsilon \tag{1.3}$$

The string overlap is used to construct the vertex operators. For instance, the general coupling of $n$ strings $\int \Psi_1 \star \Psi_2 \star \cdots \star \Psi_N$ is defined in terms of a vertex such that the left-hand side of string $i$ equates with the right-hand side of string $i+1$. Groos and Jericki (1987$a$–$c$) gave extensive results for the open-bosonic string and superstring and proved the gauge invariance by using conformal field theory.

In this paper, based on the Hilbert space of first-quantized creation and annihilation operators for the (2, 0) heterotic superstring with Wess–Zumino (WZ) term coupling the fiber bundle, we construct a superstring field theory and prove its gauge invariance by using conformal field theory.

The organization of this paper is as follows: In Section 2 the string and overlap conditions are given. In Section 3 the identity operator and three-string interacting vertex are derived. In Section 4 the reparametrization invariance and BRST symmetry of this theory are proved.

## 2. THE STRING CONSTRUCTION AND OVERLAP

The motion space of strings is assumed as a direct product of $d$-dimensional Minkowski space $M_d$ with a group manifold of $d_G$ dimensions, i.e., $M_d \otimes G$. In Xu *et al.* (n.d.) (hereafter referred to as I), we derive the (2, 0) heterotic string coupling fiber bundle. The Lagrangian takes the form (with the same notation as in I)

$$L = L_1 + L_F \tag{2.1a}$$

$$L_1 = L_\alpha + L_{WZ} \tag{2.1b}$$

$$\begin{aligned} e_{-1}L_1 = {} & -\frac{1}{2}(g^{\mu\nu}g_{\alpha\bar{\beta}} + kg^{\mu\nu}b_{\alpha\bar{\beta}})\,\partial_\mu\varphi^\alpha\,\partial_\nu\varphi^{\bar{\beta}} - i\frac{1}{2}g_{\alpha\bar{\beta}}\bar{x}^x r^\alpha(D_\mu\bar{x}_\beta \\ & + \bar{x}_\beta\gamma^\nu\gamma^\mu\Psi_\nu\,\partial_\mu\varphi^{\bar{\beta}} + \bar{x}_\beta\gamma^\mu\gamma^\mu\Psi_\nu\,\bar{\Psi}_\mu\chi^\beta) - 2iT_{\alpha\beta\bar{\gamma}}\bar{\Psi}_\beta x^\alpha\overline{x^\gamma}T^\mu x^\beta + \text{h.c.}) \end{aligned} \tag{2.1c}$$

$$\begin{aligned} e^{-1}L_F = {} & \left[\frac{1}{2}i\bar{\Psi}^A g^\mu_u\gamma^\alpha(\partial_\mu\Psi^B + A^B_{aC}\,\partial_\mu\varphi^\alpha + A^A_{aC}\,\partial_\mu\varphi^{\bar{\alpha}})\Psi^C + \text{h.c.}\right] \\ & \times G_{AB}(\varphi) - 2F_{\alpha\beta AB}\bar{x}^{\bar{\alpha}}\,\partial_\mu\chi^B\bar{\Psi}^A\gamma^\mu\psi^B \end{aligned} \tag{2.1d}$$

The heterotic superstring model can be described by the set of $x(\sigma)$, $\lambda(\sigma)$, $\xi(\sigma)$, $\chi(\sigma)$, $\psi(\sigma)$, and their conserved currents $J^{\alpha}(\sigma)$ with conformal dimensions 0, $\frac{1}{2}$, $\frac{1}{2}$, 1, $\frac{1}{2}$, $\frac{1}{2}$, and 1, respectively. The commutation relations among their modes read

$$[q^{i}, \bar{p}^{j}] = i\delta^{ij} \tag{2.2a}$$

$$[\alpha^{i}_{m}, \bar{\alpha}^{j}_{n}] = m\delta_{m+n,0}\delta^{ij} \tag{2.2b}$$

$$[\beta^{\alpha}_{m}, \bar{\beta}^{b}_{n}] = i\left(\frac{1}{k}\right)^{1/2} f^{abc}\beta^{C}_{m+n} + n\delta_{m+n,0}\delta^{ab} \tag{2.2c}$$

$$\{S^{n}_{m}, \bar{S}^{b}_{n}\} = \delta^{ab}\delta_{m+n,0} \tag{2.2d}$$

$$\{t^{i}_{m}, \bar{t}^{j}_{n}\} = \delta^{ij}\delta_{m+n,0} \tag{2.2e}$$

$$\{u^{i}_{n}, \bar{u}^{j}_{n}\} = \delta^{ij}\delta_{m+n,0} \tag{2.2f}$$

where $f^{abc}$ are group constructive constants.

To perform the quantization in the BRST scheme, one has to introduce a set of canonical anticommuting FP ghosts $c(\tau, \sigma)$, $\bar{c}(\tau, \sigma)$ as well as a set of canonical commuting FP ghosts $e(\tau, \sigma)$, $\bar{e}(\tau, \sigma)$ with conformal dimensions 2, $-1$, and 3/2, $-1/2$ respectively. The commutation relations read

$$[c_{m}, \bar{c}_{n}] = \delta_{m+n,0} \tag{2.3a}$$

$$[e_{m}, \bar{e}_{n}] = \delta_{m+n,0} \tag{2.3b}$$

We are now in a position to derive the overlap equation for the above fields. For an $N$-string vertex $|V_{N}\rangle$, we have the overlap equations

$$x(\sigma) = x^{r-1}(\pi - \sigma), \qquad P(\sigma) = P^{r-1}(\pi - \sigma) \tag{2.4a}$$

$$c^{r}(\sigma) = c^{r-1}(\pi - \sigma), \qquad \bar{c}^{r}(\sigma) = \bar{c}^{r-1}(\pi - \sigma) \tag{2.4b}$$

$$J_{r}(\sigma) = -J^{r}(\pi - \sigma), \qquad (0 \leq \sigma \leq \pi/2) \tag{2.4c}$$

$$\gamma = (1, 2, 3, \ldots, N)$$

$$A^{r}_{\pm}(\sigma) = \begin{cases} \pm A^{r-1}_{\pm}(\pi - \sigma) & \text{(NS string)} \\ +iA^{r-1}_{\pm}(\pi - \sigma) & \text{(R string)} \end{cases} \tag{2.4d}$$

A field may be any one of the $\chi$, $\xi$, $\psi$, $e$ fields,

$$\bar{e}^{r}(\pi - \sigma) = \begin{cases} i\bar{e}^{r-1}(\pi - \sigma) & \text{(NS)} \\ i\bar{e}^{r-1}(\pi - \sigma) & \text{(R)} \end{cases} \tag{2.5}$$

For the Virasoro generators, we have

$$L^{r}(\sigma) = L^{r-1}(\pi - \sigma), \qquad 0 \leq \sigma \leq \pi \tag{2.6}$$

It is readily known from (2.6) that the operator $K_n = L_n - (-1)^n L_{-n}$ would annihilate $|I\rangle$. In other words, $K_n$ will be the derivative corresponding to the integration of (1.1a). The overlap equations for an identity operator $|I\rangle$ can thus be obtained similarly.

## 3. THE DERIVATION OF IDENTITY OPERATOR AND THREE-STRING INTERACTION VERTEX

We first consider the explicit expression of the identity operator $|I\rangle$ in terms of the overlap equation for a single string. In the case of a field having integer conformal dimension, the identity operators $|I^x\rangle$, $|I^{gh}\rangle$, $|I^J\rangle$ have the overlap equations

$$[x(\sigma) - x(\pi - \sigma)]|I^x\rangle = 0 \tag{3.1a}$$

$$[p(\sigma) + p(\pi - \sigma)]|I^x\rangle = 0 \tag{3.1b}$$

$$[c(\sigma) + c(\pi - \sigma)]|I^{gh}\rangle = 0 \tag{3.1c}$$

$$[\bar{c}(\sigma) + \bar{c}(\pi - \sigma)]|I^{gh}\rangle = 0 \tag{3.1d}$$

$$[J(\sigma) + J(\pi - \sigma)]|I^J\rangle = 0 \tag{3.1e}$$

or equivalently, resolved into their modes:

$$[\alpha^{\mu}_m + (-)^m \alpha^{\mu}_{-m}]|I^x\rangle = 0 \tag{3.2a}$$

$$[\bar{c}_m + (-1)^m \bar{c}_{-m}]|I^{gh}\rangle = 0 \tag{3.2b}$$

$$[c_m - (-1)^m c_{-m}]|I^{gh}\rangle = 0 \tag{3.2c}$$

$$[J^{\alpha}_m + (-1)^m J^{a}_{-m}]|I^J\rangle = 0 \tag{3.2d}$$

One may then write $|I^x\rangle$ and $|I^{gh}\rangle$ in the Gaussian form of creation operators. From (3.1), we are led to unique expressions for them:

$$|I^x\rangle = \exp\left\{-\frac{1}{2}\sum_{n=1}^{m}(-1)^n \frac{1}{n}\alpha^n_{-n}\alpha^n_{-n}\right\}|0\rangle \tag{3.3a}$$

$$|I^{gh}\rangle = \exp\left\{\sum_{n=1}^{m}(-1)^n e_{-n}\bar{e}_{-n}\right\}|0\rangle_{1/2} \tag{3.3b}$$

where $|0\rangle_{1/2} = |c_0 = 0\rangle$. For the field having a half-integer conformal dimension, the Gaussian expression of its identity operator can be found by using the Neumann function via conformal mapping (Gross and Jericki, 1987*a*–*c*). Passing over the computation of the correlation function, we may give the Gaussian form of the creation operator for the identity operator as

(consider the Nevieu–Schwarz string only)

$$|I^{\lambda}\rangle = \exp\left\{\frac{1}{2}\sum_{m,n=1/2}^{\infty} d^{\alpha}_{m} I_{mn} d^{\alpha}_{n}\right\}|0\rangle \tag{3.4a}$$

$$|I^{\chi}\rangle = \exp\left\{\frac{1}{2}\sum_{m,n=1/2}^{\infty} S^{\alpha}_{m} I_{mn} S^{\alpha}_{-n}\right\}|0\rangle \tag{3.4b}$$

$$|I^{\xi}\rangle = \exp\left\{\frac{1}{2}\sum_{m,n=1/2}^{\infty} t^{\alpha}_{-m} I_{mn} t^{\alpha}_{-n}\right\}|0\rangle \tag{3.4c}$$

$$|I^{\psi}\rangle = \exp\left\{\frac{1}{2} u^{\alpha}_{-m} I_{mn} u^{\alpha}_{-n}\right\}|0\rangle \tag{3.4d}$$

$$|I^{sgh}\rangle = \exp\left\{\sum_{m,n=1/2}^{\infty} \beta_{-m}\bar{I}\gamma_{-n}\right\}|0\rangle \tag{3.4e}$$

respectively, where the Neumann functions are

$$I(p, p') = \left(\frac{\partial\omega}{\partial\rho}\right)^{1/2} \frac{1}{(\omega - \omega')} \left(\frac{\partial\omega'}{\partial\rho'}\right)^{1/2} \tag{3.5a}$$

$$\overline{I(\rho, \rho')} = \frac{1}{2}\left[\frac{z}{z'} + \frac{z'}{z}\right]\left(\frac{\partial\omega'}{\partial\rho'}\right)^{1/2} \frac{1}{(\omega - \omega')} \left(\frac{\partial\omega}{\partial\rho}\right)^{1/2} \tag{3.5b}$$

The corresponding quadratic forms are

$$\Delta^{\lambda} = \int_{\pi}^{\pi} d\sigma \frac{1}{2\pi} \int_{\pi}^{\pi} d\sigma' \frac{1}{2\pi} \lambda^{\alpha}_{cr}(\sigma) I^{+}(\sigma, \sigma') \lambda^{\alpha}_{cr}(\sigma') \tag{3.6a}$$

$$\Delta^{\chi} = \int_{-\pi}^{\pi} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma' \frac{1}{2\pi} \chi^{\alpha}_{cr}(\sigma) I^{+} \chi^{\alpha}_{cr}(\sigma') \tag{3.6b}$$

$$\Delta^{\xi} = \int_{-\pi}^{\pi} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma' \frac{1}{2\pi} \xi^{\alpha}_{cr}(\sigma) I^{+}(\sigma, \sigma') \xi^{\alpha}_{cr}(\sigma') \tag{3.6c}$$

$$\Delta^{\psi} = \int_{-\pi}^{\pi} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma' \frac{1}{2\pi} \psi^{\alpha}_{cr}(\sigma) I^{+} \psi^{\alpha}_{cr}(\sigma') \tag{3.6d}$$

where the superscript + denotes the creation operator parts. The case of constructing three-vertex operators are similar. Through the same manipulation, their explicit forms read

$$|V^{x}_{3}\rangle = \exp\left\{\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=1}^{\infty} \alpha^{(r)}_{-m} K^{rg}_{mn} \alpha^{a(r)}_{-n}\right\}|0\rangle \tag{3.7a}$$

$$|V^{\lambda}_{3}\rangle = \exp\left\{\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=1/2}^{\infty} d^{a(r)}_{-m} K^{rg}_{mn} d^{a(r)}_{-a}\right\}|0\rangle \tag{3.7b}$$

$$|V_3^x\rangle = \exp\left\{\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=1/2}^{\infty} S_{-m}^{a(r)} K_{mn}^{rg} S_{-n}^{a(r)}\right\}|0\rangle \tag{3.7c}$$

$$|V_3^{\xi}\rangle = \exp\left\{\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=1/2}^{\infty} t_{-m}^{a(r)} K_{mn}^{rg} t_{-n}^{a(r)}\right\}|0\rangle \tag{3.7d}$$

$$|V_3^{\xi}\rangle = \exp\left\{\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=1/2}^{\infty} u_{-m}^{a(r)} K_{mn}^{rg} u_{-n}^{a(r)}\right\}|0\rangle \tag{3.7e}$$

Here the Neumann functions are given by

$$K^x(p, p) = In(\omega - \omega') \tag{3.8a}$$

$$K^{\lambda}(p, p') = K^{\chi}(\omega, \omega') = K^{\xi}(p, p') = K^{\psi}(p, p') = \left(\frac{\partial \omega}{\partial \rho}\right)^{1/2} \frac{1}{\omega - \omega'} \left(\frac{\partial \omega'}{\partial \rho'}\right)^{1/2} \tag{3.8b}$$

$$K^J(p, p') = \frac{\partial \omega}{\partial \rho} \frac{1}{(\omega - \omega')^2} \frac{\partial \omega'}{\partial \rho'} \tag{3.8c}$$

and the quadratic form is given by

$$\Delta^A = \int_{-\pi}^{\pi} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma' \frac{1}{2\pi} A_{cy}^r(\sigma)[K_{ry}^{(A)}(\sigma, \sigma')]^+ A_{cy}^r(\sigma') \tag{3.9}$$

where $A$ is any one of the fields $x$, $\lambda$, $\chi$, $\xi$, $\psi$. Vertex operators for conformal and superconformal ghost fields are given by

$$|V_3^{gh}\rangle = \exp\left\{\sum_{r,g=1}^{3}\sum_{m,n=1}^{\infty} b_{mn}^r K_{mn}^{ghrg} n c_m^r\right\}|0\rangle \tag{3.10a}$$

$$|V_3^{rgh}\rangle = \exp\left\{\sum_{r,g=1}^{3}\sum_{m,n=1}^{\infty} \beta_{-m}^r K_{mn}^{sghrg} \gamma_{-n}^r\right\}|0\rangle \tag{3.10b}$$

Here the Neumann functions are given as follows:

$$K^{gh}(\rho, \rho') = \frac{1}{2}\left[\frac{z}{z'} + \frac{z'}{z}\right] \frac{\partial \omega'}{\partial \rho'} \frac{\partial \omega}{\partial \rho} \tag{3.11a}$$

$$K^{sgh}(\rho, \rho') = \frac{1}{2}\left[\frac{z}{z'} + \frac{z'}{z}\right] \left(\frac{\partial \omega'}{\partial \rho'}\right)^{1/2} \frac{1}{\omega - \omega'} \left(\frac{\partial \omega}{\partial \rho}\right)^{1/2} \tag{3.11b}$$

where

$$Z|_r = (\omega|_r)^{1/2} = z_r \left(\frac{1 + ie^{i\theta}}{1 - ie^{i\theta}}\right)^{1/2}, \qquad r = 1, 2, 3$$

and the quadratic forms are given by

$$\Delta^{\alpha\bar{\alpha}} = \int_{-\pi}^{\pi} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma' \frac{1}{2\pi} e^{r}_{cy}(\sigma)[K^{ghrs}(\sigma, \sigma')]^{+}\bar{e}^{r}_{cy}(\sigma') \tag{3.12a}$$

$$\Delta^{e\bar{e}} = \int_{-\pi}^{\pi} d\sigma \frac{1}{2\pi} \int_{-\pi}^{\pi} d\sigma' \frac{1}{2\pi} \beta^{r}_{cy}(\sigma)[K^{sghrs}(\sigma, \sigma')]^{+}\bar{\gamma}^{r}_{cy}(\sigma') \tag{3.12b}$$

respectively, while $|0\rangle_{3/2} = |C_0^1 = 0\rangle = |C_0^2 = 0\rangle = |C_0^3 = 0\rangle$ has ghost number 3/2.

We introduce the following modified vertices for superconformal ghosts:

$$|V_3'^{sgh}\rangle = X\left(\frac{\pi}{2}\right)e^{-\Phi(\pi/2)}|V_3^{sgh}\rangle \tag{3.13a}$$

$$|V_3'^{sgh}\rangle = Y\left(\frac{\pi}{2}\right)e^{\Phi(\pi/2)}|I^{sgh}\rangle \tag{3.13b}$$

Here, two midpoint modification factors, the picture-changing operator and the inverse picture-changing operator, are as follows:

$$X = e^{\Phi}\left\{\rho^{\alpha}\lambda^{\alpha} + \frac{1}{(1 + C_A/2k)^{1/2}}\left[J^{\alpha}\chi^{\alpha} - \frac{i}{6\sqrt{k}}\int^{\alpha bc} \chi^{\alpha}\chi^{b}\chi^{c}\right]\right\}$$
$$+ \bar{C}\,\partial\xi + e^{2\Phi}2\eta C$$
$$Y = \bar{C}\,\partial\xi\, e^{-2\Phi}$$

where $\xi$ and $\eta$ are anticommuting variables with conformal dimensions 0 and 1, respectively, while $e^{\Phi}$ and $e^{-\Phi}$, denoted by $e^{\Phi} = \delta(e)$ and $e^{-\Phi} = \delta(\bar{e})$ formally, have ghost numbers 1 and $-1$, respectively, whose conformal dimensions are 3/2 and $-1/2$, respectively.

## 4. THE QUANTIZATION OF THE STRING

In order to quantize the heterotic superstring constructed above, we introduce the BRST charge $Q$ as follows (Xu *et al.*, n.d.):

$$Q = \sum L'^{\tau}_{-s}\bar{e}^{\tau}_{s} + \sum F'^{\tau}_{m}\bar{e}^{\tau}_{s} - \frac{1}{2}\sum (m - n) :\bar{e}^{\tau}_{-s}\bar{e}^{\tau}_{-s}\bar{e}^{\tau}_{m+s}: - \sum\left(m - \frac{1}{2}n\right)$$
$$\times :\bar{e}^{\tau}_{-s}\bar{e}^{\tau}_{-s}e^{\tau}_{m-s}: - \sum :\bar{e}^{\tau}_{-s}\bar{e}^{p}_{-s}e^{\tau}_{m+s}: - \sum :\bar{e}^{\tau}_{-s}\bar{e}^{\tau}_{-s}e^{\tau}_{m+s} - \alpha c_{\alpha}: \tag{4.1}$$

where

$$L'_s = \frac{1}{2}\sum :\alpha_{s-m}^{(\alpha(p)}\alpha_m^{\alpha(\gamma)}: + \frac{1}{2}\sum\left(m - \frac{1}{2}n\right):S_{s-m}^{\alpha(p)}S_m^{\alpha(p)}: + \frac{1}{2}\sum\left(m - \frac{1}{2}n\right)$$

$$\times :t_{s-m}^{\alpha(p)}t_m^{\alpha(\gamma)}: + \frac{1}{2}\sum\left(m + \frac{1}{2}n\right):u_{s-m}^{\alpha(p)}u_m^{(\alpha(\gamma)}:$$

$$+\frac{1}{2[1 + C_A/2k]}:J_{s-m}^{\alpha(p)}J_m^{\alpha(p)}: \tag{4.2a}$$

$$F_s^{\prime\tau} = \sum :\alpha_{s-m}^{\alpha(p)}d_m^{\alpha(p)}: + \sum\frac{1}{(1 + C_A/2k)^{1/2}}\left\{\sum :J_{s-m}^{\alpha(p)}S_m^{\alpha(p)}:\right.$$

$$\left. - \frac{1}{(\sigma\sqrt{K})}\int^{\alpha bc}\sum :\beta_t^{\alpha(\tau)}\beta_{s-t-m}^{\alpha(p)}S_s^{\alpha(\tau)}: + \sum :\beta^{\alpha(\tau)}u^{\alpha(\tau)}: + \sum :S^{\alpha(\tau)}u^{\alpha(r)}:\right\} \tag{4.2b}$$

$$K_s = L_s - (-1)^s L_{-s} \tag{4.2c}$$

We can prove that $K_n$ and $Q$ are mutually related as follows:

$$K_n = \left\{\sum_{\tau=1}^{D}[e_n^\tau - (-)^n e_n^\tau], Q\right\} = \sum_{\tau=1}^{D}[L_n^\tau - (-1)^n L_n^\tau] = L_n - (-1)^n L_{-n} \tag{4.3a}$$

$$F_n = \left[Q, \sum_{\tau=1}^{D} e_n^\tau\right] = F'_n - \sum\left(n - \frac{1}{2}m\right):c_{-n}\beta_{n+m}: - 2\sum :\bar{e}_{-m}c_{n+m}: \tag{4.3b}$$

where

$$L_n = L'_n + \sum(n+m):\bar{c}_{n-m}^\tau\bar{c}_m^\tau: + \sum\left(\frac{1}{2}n + m\right):e_{n-m}^\tau\bar{e}_m^\tau: - D\alpha\delta_{n,0}$$

$$= L'_n + L_n^{gh} + L_n^{\tau gh} + D\alpha\delta_{n,0} \tag{4.3c}$$

with the $D$ representing the number of strings. For the identity operator, $D = 1$, while for the three-string vertex, $D = 3$.

We are now in a position to demonstrate the reparametrization and BRST invariance of the vertex operator (Green *et al.*, 1987; Trami, 1991), namely

$$K_n|V_D\rangle = 1 \tag{4.4a}$$

$$Q|V_D\rangle = 0, \qquad D = 1, 2, \ldots, N \tag{4.4b}$$

By an algebraically tedious (but simple) calculation, one can prove that

$$K^{x}_{2N}|I^{x}\rangle = -\frac{d}{2}N(-)^{N}|I^{x}\rangle \tag{4.5a}$$

$$K^{gh}_{2N}|I'^{gh}\rangle = \frac{26}{2}N(-)^{N}|I'^{gh}\rangle \tag{4.5b}$$

$$K^{j}_{2N}|I^{J}\rangle = -\frac{d_G}{2}\frac{1}{[1+C_A/2k]}N(-)^{N}|I^{J}\rangle \tag{4.5c}$$

$$K^{J}_{2N}|V^{J}_{3}\rangle = \frac{1}{2[1+C_A/2k]}\sum_{\tau=1}^{3}\sum_{n=1}^{2N-1}(2N-m)mK^{(J)pp}_{m,2N-m}|V^{J}_{3}\rangle \tag{4.5d}$$

$$K^{\xi}_{2N}|V^{\xi}_{3}\rangle = N(-)^{N}\frac{5}{2\cdot 3^{2}}\frac{d}{2}|V^{\xi}_{3}\rangle \tag{4.5c}$$

$$K^{\Psi}_{2N}|V^{\Psi}_{3}\rangle = N(-)^{N}\frac{5}{2\cdot 3^{2}}\frac{1}{2}|V^{\Psi}_{3}\rangle \tag{4.5f}$$

$$\vdots$$

In conclusion, we obtain the following results:

$$\begin{aligned} K_{2N}|I\rangle &= K_{2N}Y(\tfrac{n}{2})\delta(e(\tfrac{n}{2}))|I^{n}\rangle|I^{\lambda}\rangle|I^{\xi}\rangle|I^{J}\rangle|I'^{gh}\rangle|I'^{sgh}\rangle \\ &= k_{2N}N(-1)^{N}|I\rangle \end{aligned} \tag{4.6a}$$

$$\begin{aligned} K_{2N}|V_3\rangle &= k'_{2N}X(\tfrac{n}{2})e^{-\Phi}|V^{x}_{3}\rangle|V^{\lambda}_{3}\rangle|V^{\xi}_{3}\rangle|V^{\chi}_{3}\rangle|V^{J}_{3}\rangle|V'^{gh}_{3}\rangle|V'^{sgh}_{3}\rangle \\ &= k'_{2N}N(-1)^{N}|V_3\rangle \end{aligned} \tag{4.6b}$$

where

$$\begin{aligned} k_{2N} &= -4 + k^{x}_{2N} + k^{\lambda}_{2N} + k^{J}_{2N} + k^{\chi}_{2N} + k^{\xi}_{2N} + k^{\Phi}_{2N} + k^{gh}_{2N} + k^{sgh}_{2N} \\ &= \left\{-4 - \frac{d}{2} - \frac{dG}{4} - \frac{dG}{5} - \frac{4}{15}\frac{dG}{1+C_A/2k} - \frac{dG}{2}\frac{1}{[1+C_A/2k]}\right. \\ &\quad \left. - \frac{d}{4} - \frac{d}{4} - \frac{1}{4} + 13 - \frac{3}{2}\right\}N(-1)^{N^{p}} \end{aligned} \tag{4.6c}$$

$$\begin{aligned} k'_{2N} &= 4 + \frac{5}{2\cdot 3}\left\{d + \frac{d_G}{[1+C_A/2k]} + \frac{d}{2} + \frac{d_G}{2} + \frac{1}{5}dG + \frac{4}{15}\frac{dG}{1+C_A/2k}\right. \\ &\quad \left. + \frac{d}{2} + \frac{1}{2} - 26 - \frac{17}{5}\right\} \end{aligned} \tag{4.6d}$$

It follows from (4.6c) and (4.6d) that when

$$d = 2 - \frac{2}{15} dG - \frac{4}{15} \frac{d_G}{1 + C_A/2k} \tag{4.7a}$$

$$I = 22 - \frac{7}{15} dG - \frac{2}{5} \frac{d_G}{1 + C_A/2k} \tag{4.7b}$$

$k_{2N}$, $k'_{2N}$ vanish. These results are the same as Xu *et al.* (n.d.) and show the reparametrization invariance. For the other boundary conditions we get a similar result. From

$$Q|I\rangle = \sum_{s=1}^{\infty} \bar{e}_{-s} k_s |I\rangle \tag{4.8a}$$

$$Q|V_3\rangle = \frac{1}{3} \sum_{\tau=1}^{3} \sum_{s=1}^{\infty} \bar{e}^{\tau}_{-s} k_s |V_3\rangle \tag{4.8b}$$

we may infer the BRST invariance.

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